Application of Hypersingular Integral Equation Method to a Three-Dimensional Crack in Piezoelectric Materials*

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Using the Green's functions, the general solutions of a three-dimensional crack problem in piezoelectric materials under mechanical and electrical loads is derived by boundary element method. Then this crack problem is reduced to solve a set of hypersingular integral equations coupled with boundary integral equations. The unknown functions are the discontinuities of the elastic displacements and electrical potential of the crack surface. The singularity of the unknowns at the crack front is analyzed by the main-part analysis method of two-dimensional hypersingular integral equations, and the exact analytical solution of the singular stresses and electrical displacements near the crack front in a transversely isotropic piezoelectric solid is given.

Key Words: Piezoelectric, Crack, Body Force Method, Boundary Element Method, Hypersingular Integral Equation

1. Introduction

The piezoelectric materials have coupled effects between the elastic and the electric fields, and have become of major interest as the functional materials such as actuators and sensors. It is possible to make a system of intelligent composite materials by combining these piezoelectric materials with structural materials. On the other hand, both electrical and mechanical disturbances are present in piezoelectric components, and the strength of the piezoelectric materials is weakened by the presence of defects such as voids and cracks. The reliability of these structures depends on the knowledge of applied mechanical and electric disturbances. When cracks are present, they may grow under service load and affect the performance of structures. Due to the disadvantage of brittleness and low fracture toughness of piezoelectric materials, a considerable number of research works have been carried out to investigate the fracture behavior(1)–(14).

Because of mathematical difficulties to treat the coupled electromechanical fields in piezoelectricity, the majority of the literature concerning crack problems is based on two-dimensional assumptions. Comparatively, few exact solutions are available in the literature for three-dimensional crack problems in piezoelectric materials. A solution of a penny-shaped impermeable crack subjected to axisymmetric tensile loading was derived by Wang(15) using a proposed general potential function approach. Using two potential functions, Wang and Huang(16) obtained the solution for an elliptical crack under uniform tractions and electric disturbance, if the plane of transversal isotropy is parallel to the crack. Chen and Shioya(17) developed the complete and exact solutions of a penny-shaped crack in a piezo-electric solid for shear loadings by a potential theory. Closed-form solutions for other 3D crack configurations in an infinite piezoelectric body are yet to be found. Thus, to assess crack-like defects in piezoelectric materials under combined mechanical and electric loadings more efficiently, it is necessary to establish appropriate numerical tools. There are two important numerical methods. One is the finite element method (FEM), and another is the boundary element method (BEM). Kuna and Shang(8),(9) have analyzed penny-shaped and elliptical cracks subjected to combined mechanical tension and electric fields by FEM, and presented some numerical results of the stress-intensity factors and energy release rates. BEM is a powerful tool for the solution of field problems of mathematical physics, since it offers some inherent advantages over FEM, like the discretization of the boundary only and an improved accuracy in flux cal-

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culations. Many publications have already been devoted to the development of fundamental solutions and BEM for piezoelectricity[(1),(18),(19)], but only a very limited number of them deals with three-dimensional analyses, due to the problems involved resulting from the anisotropy of piezoelectric materials. A 3D Green's function for static piezoelectricity and its derivatives have been presented by Deeg[(1)] for piezoelectrics of general anisotropy. Dynamic piezoelectric Green's functions have been presented by Norris[(6)] in the frequency domain and by Khutoryansky and Sosa[(20),(21)] in the time domain. For the particular case of transversely isotropic piezoelectricity, Dunn and Wienecke[(22)] for piezoelectrostatics, and Daros and Antes[(23)] for transient analysis developed simplified expressions for the Green's functions. BEM for static piezoelectricity with corresponding numerical results for 3D analysis has been presented in 1995 by Chen and Lin[(24)], and in 1998 by Hill and Farris[(11)].

The hypersingular integral equation based on the body force method is a good analytic and numerical method in fracture mechanics, and has been widely used for fracture mechanics[(25)−(27)]. In this paper, the Green's functions are used to derive the general solutions of a three-dimensional crack in piezoelectric materials under mechanical and electrical loads by boundary element method. The crack problem is reduced to solve a set of hypersingular integral equations coupled with boundary integral equations. The singularities of the elastic displacements and electrical potential near the crack front are analyzed by the main-part analysis method, and the exact analytical solution of the singular stresses and electrical fluxes are given.

2. Basic of Piezoelectricity

The linear governing equations and constitutive relations for a piezoelectric material in static equilibrium can be expressed as two separate equations, one representing conservation of momentum and the other conservation of electric charge[(1)−(5)]. To use these two equations in conjunction with the developed boundary integral equation method, they are combined into one. In these equations, lowercase indices i can have values of 1, 2, or 3, and uppercase indices I can take on values of 1, 2, 3, and 4. The modified governing equation for the piezoelectric material in static equilibrium can be written as[(1)]

$$\Sigma_{iJ,i} + b_J = 0 \tag{1}$$

where Σ_{iJ} is the stress-electric displacement matrix, defined as

$$\Sigma_{iJ} = \begin{cases} \sigma_{ij} & \text{for } J = j = 1,2,3 \\ D_i & \text{for } J = 4 \end{cases} \tag{2}$$

and b_J is the body load (force and charge) column vector. A subscript comma denotes the partial differentiation. The combined constitutive equation is written as

$$\Sigma_{iJ} = E_{iJKl} Z_{Kl} \tag{3}$$

where E_{iJKl} is the electroelastic constant matrix

$$E_{iJKl} = \begin{cases} c_{ijkl} & \text{for } J,K = 1,2,3 \\ e_{lij} & \text{for } J = 1,2,3, \quad K = 4 \\ e_{ikl} & \text{for } J = 4 \quad K = 1,2,3, \\ -a_{il} & \text{for } J = 4 \quad K = 4 \end{cases} \tag{4}$$

and the strain-electric field matrix Z_{Kl} takes the form

$$Z_{Kl} = \begin{cases} \varepsilon_{kl} & \text{for } K = k = 1,2,3 \\ \phi_{,l} & \text{for } K = 4 \end{cases} \tag{5}$$

In addition, U_K is the elastic displacement-electric potential matrix

$$U_K = \begin{cases} u_k & \text{for } K = k = 1,2,3 \\ \phi & \text{for } K = 4 \end{cases} \tag{6}$$

where u_k and ϕ are the elastic displacement and electric potential, respectively.

3. Boundary Integral Equations for a Crack in General Piezoelectric Materials

3.1 Boundary condition of a crack surface

The mechanical boundary condition of cracks in piezoelectric materials is always defined by stress-free crack surfaces. Several electric boundary conditions were proposed in literature. Among these electric boundary conditions, two different conditions are applied widely. Those are permeable and impermeable conditions. For the first one, the normal electric displacement and electric potential should be continuous across the crack surface

$$D_3^+ = D_3^- \quad \phi^+ = \phi^- \tag{7}$$

where the superscripts + and − denote the upper and lower crack surfaces, respectively. This aspect has been supported by McMeeking[(12)], and Dunn[(22)]. Pak[(2)], and Suo, et al.[(4)] proposed impermeable conditions on the crack faces

$$D_3^+ = D_3^- = 0 \tag{8}$$

This paper presents an analysis for the crack problems in piezoelectric materials based on boundary condition (8).

3.2 General solutions for a crack in a three-dimensional piezoelectric solid

Consider a flat crack S in a finite three-dimensional piezoelectric solid. A fixed rectangular Cartesian system x_i ($i = 1,2,3$) is used. The crack is assumed to be in the x_1x_2 plane, and normal to the x_3 axis. Using the piezoelectric form of the Somigliana identity, the elastic displacements and the electric potential at an interior point p are expressed as[(11)]

$$U_I(p) = -\int_\Gamma T_{IJ}(p,Q) U_J(Q) ds(Q)$$

$$+\int_{\Gamma} U_{IJ}(p,Q)T_J(Q)ds(Q)$$
$$-\int_{S^+ + S^-} T_{IJ}(p,Q)U_J(Q)ds(Q)$$
$$+\int_{S^+ + S^-} U_{IJ}(p,Q)T_J(Q)ds(Q)$$
$$+\int_{\Omega} U_{IJ}(p,Q)b_J(Q)d\Omega(Q) \quad I,J=1,2,3,4 \tag{9}$$

where Ω is the domain occupied by the piezoelectric solid, Γ is the external boundary, T_J are the elastic tractions and normal charge flux densities on the boundary, U_{IJ} and T_{IJ} are the fundamental solutions of the piezoelectric material and related Green's function as follows[11]

$$U_{IJ}(\xi - x) = G_{IJ}(\xi - x) \tag{10}$$

$$T_{IJ}(\xi - x) = E_{kJMn}\frac{\partial G_{IM}(\xi - x)}{\partial \xi_n} \tag{11}$$

$$T_J = \begin{cases} t_j = \sigma_{jl}n_l & \text{for } J=j=1,2,3 \\ q = D_l n_l & \text{for } J=4 \end{cases} \tag{12}$$

Note that the elastic displacement and electric potential discontinuities are written as

$$\tilde{U}_J = \begin{cases} \tilde{u}_j = u_j^+ - u_j^- & \text{for } J=j=1,2,3 \\ \tilde{\phi} = \phi^+ - \phi^- & \text{for } J=4 \end{cases} \tag{13}$$

Using the relations $T_{IJ}(p,Q^+) = -T_{IJ}(p,Q^-) = T_{IJ}^+(p,Q)$ and $U_{IJ}(p,Q^+) = U_{IJ}(p,Q^-)$, the elastic displacements and the electric potential (9) can be rewritten as

$$U_I(p) = -\int_{\Gamma} T_{IJ}(p,Q)U_J(Q)ds(Q)$$
$$+\int_{\Gamma} U_{IJ}(p,Q)T_J(Q)ds(Q)$$
$$-\int_{S^+} T_{IJ}^+(p,Q)\tilde{U}_J(Q)ds(Q)$$
$$+\int_{\Omega} U_{IJ}(p,Q)b_J(Q)d\Omega(Q)$$
$$I,J=1,2,3,4 \tag{14}$$

Using solution (14) and constitutive Eq. (3), the corresponding stress and electric displacements are expressed as

$$\Sigma_{iJ}(p) = -\int_{\Gamma} S_{KiJ}(p,Q)U_K(Q)ds(Q)$$
$$+\int_{\Gamma} D_{KiJ}(p,Q)T_K(Q)ds(Q)$$
$$-\int_{S^+} S_{KiJ}^+(p,Q)\tilde{U}_K(Q)ds(Q)$$
$$+\int_{\Omega} D_{KiJ}(p,Q)b_K(Q)d\Omega(Q) \tag{15}$$

where the integral kernels are as follows

$$S_{KiJ}(p,Q) = E_{iJMn}\frac{\partial T_{MK}(p,Q)}{\partial x_n} = -E_{iJMn}\frac{\partial T_{MK}(p,Q)}{\partial \xi_n} \tag{16}$$

$$D_{KiJ}(p,Q) = E_{iJMn}\frac{\partial U_{MK}(p,Q)}{\partial x_n} = -E_{iJMn}\frac{\partial U_{MK}(p,Q)}{\partial \xi_n} \tag{17}$$

3.3 Boundary integral equations

For a finite piezoelectric solid with an embedded flat crack, there are two parts of boundary. One is the external boundary Γ, and another is the crack surface $S^{\pm}$. Using the boundary conditions, the boundary integral equation and hypersingular integral equations can be obtained. Let the source point p be taken to the boundary Γ and represented by P, the boundary integral equation can be derived from Eq. (14) as follows

$$C_{IJ}U_J(P) + \int_{\Gamma} T_{IJ}(P,Q)U_J(Q)ds(Q)$$
$$+\int_{S^+} T_{IJ}^+(P,Q)\tilde{U}_J(Q)ds(Q)$$
$$=\int_{\Gamma} U_{IJ}(P,Q)T_J(Q)ds(Q)$$
$$+\int_{\Omega} U_{IJ}(P,Q)b_J(Q)d\Omega(Q) \qquad P \in \Gamma \tag{18}$$

where C_{IJ} is the constant related to the boundary point P, which is not evaluated directly.

Using the elastic and electric boundary conditions of the crack surface, the hypersingular integral equations can be obtained as

$$\neq\!\!\!\!\!\!\int_{S^+} S_{KiJ}^+(P,Q)\tilde{U}_K(Q)ds(Q)$$
$$+\int_{\Gamma} S_{KiJ}(P,Q)U_K(Q)ds(Q)$$
$$=\int_{\Gamma} D_{KiJ}(P,Q)T_K(Q)ds(Q)$$
$$+\int_{\Omega} D_{KiJ}(P,Q)b_K(Q)d\Omega(Q) \quad P \in S^+ \tag{19}$$

where $\neq\!\!\!\!\!\!\int$ means that the integral must be interpreted as a finite-part integral. The first integral in Eq. (19) has the order r^{-3}, and is a hypersingular one. Solving Eqs. (18) and (19), all the unknowns can be obtained.

4. A Crack in an Infinite Transversely Isotropic Piezoelectric Solid

Consider a flat crack embedded in an infinite transversely isotropic piezoelectric solid. Suppose that the crack surface is parallel to the symmetric plane (e.g. x_1x_2 plane), and there are no body forces and charge. For the transversely isotropic piezoelectric material, the electro-elastic constants can be written as follows

$$c_{ijkl} = c_{12}\delta_{ij}\delta_{kl} + c_{66}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$
$$+(c_{13} - c_{12})(\delta_{ij}\delta_{3k}\delta_{3l} + \delta_{3i}\delta_{3j}\delta_{kl})$$

$$+(c_{44}-c_{66})(\delta_{jk}\delta_{3i}\delta_{3l}+\delta_{ik}\delta_{3j}\delta_{3l}+\delta_{il}\delta_{3j}\delta_{3k}$$
$$+\delta_{jl}\delta_{3i}\delta_{3k})+(c_{11}+c_{33}-2c_{13}-4c_{44})\delta_{3i}\delta_{3j}\delta_{3k}\delta_{3l} \quad (20)$$

$$e_{lij}=e_{31}\delta_{ij}\delta_{3l}+e_{15}(\delta_{il}\delta_{3j}+\delta_{jl}\delta_{3i})$$
$$+(e_{33}-c_{31}-2e_{15})\delta_{3i}\delta_{3j}\delta_{3l} \quad (21)$$

$$a_{il}=a_{11}\delta_{il}+(a_{33}-a_{11})\delta_{3i}\delta_{3l} \quad (22)$$

here $c_{66}=(c_{11}-c_{12})/2$.

4.1 Green's solution

For transversely isotropic piezoelectric, Green's function can be written as an explicit expression. Here we use the solutions given by Dunn and Wienecke[22] by a potential method. The governing equations are expressed as

$$\begin{cases} u_1=\left\{(c_{13}e_{15}-c_{44}e_{31})\dfrac{\partial^2}{\partial x_1\partial x_3}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)+[(c_{44}+c_{13})e_{33}-c_{33}(e_{15}+e_{31})]\dfrac{\partial^4}{\partial x_1\partial x_3^3}\right\}g-\dfrac{\partial\psi}{\partial x_2} \\ u_2=\left\{(c_{13}e_{15}-c_{44}e_{31})\dfrac{\partial^2}{\partial x_2\partial x_3}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)+[(c_{44}+c_{13})e_{33}-c_{33}(e_{15}+e_{31})]\dfrac{\partial^4}{\partial x_2\partial x_3^3}\right\}g+\dfrac{\partial\psi}{\partial x_1} \\ u_3=\left\{-c_{11}e_{15}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)^2-c_{44}e_{33}\dfrac{\partial^4}{\partial x_3^4}+[c_{13}(e_{15}+e_{31})+c_{44}e_{31}-c_{11}e_{33}]\dfrac{\partial^2}{\partial x_3^2}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)\right\}g \\ \phi=\left\{c_{44}c_{11}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)^2+c_{44}c_{33}\dfrac{\partial^4}{\partial x_3^4}+(c_{11}c_{33}-2c_{44}c_{13}-c_{13}^2)\dfrac{\partial^2}{\partial x_3^2}\left(\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}\right)\right\}g \end{cases} \quad (23)$$

where the potentials g and ψ must satisfy following equations:

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{\nu_1^2}\frac{\partial^2}{\partial x_3^2}\right)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{\nu_2^2}\frac{\partial^2}{\partial x_3^2}\right)\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{\nu_3^2}\frac{\partial^2}{\partial x_3^2}\right)g=0 \quad (24)$$

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{1}{\nu_0^2}\frac{\partial^2}{\partial x_3^2}\right)\psi=0 \quad (25)$$

here $\nu_0=\sqrt{c_{66}/c_{44}}$, and $-1/\nu_1^2$, $-1/\nu_2^2$, $-1/\nu_3^2$ are the roots of the following cubic equation

$$s^3+\frac{a}{d}s^2+\frac{b}{d}s+\frac{c}{d}=0 \quad (26)$$

where

$$\begin{cases} a=c_{11}(a_{11}c_{33}+2e_{15}e_{33})-a_{11}c_{13}(c_{13}+2c_{44})+c_{44}(a_{33}c_{11}+e_{31}^2)-2e_{15}c_{13}(e_{31}+e_{15}) \\ b=c_{33}[a_{11}c_{44}+a_{33}c_{11}+e_{31}(e_{31}+e_{15})]-c_{13}a_{33}(c_{13}+2c_{44})+(e_{15}+e_{31})(c_{33}e_{15}-2c_{13}e_{33}) \\ \quad +e_{33}(c_{11}e_{33}-2c_{44}e_{31}) \\ c=c_{44}(a_{33}c_{33}+e_{33}^2) \\ d=c_{11}(a_{11}c_{44}+e_{15}^2) \end{cases} \quad (27)$$

If the above Eqs. (24) and (25) are solved for a point charge or force, the Green's functions can be obtained from the solutions u_i and ϕ.

4.1.1 Point force charge For a unit point charge at point $\boldsymbol{\xi}(\xi_1,\xi_2,\xi_3)$, the elastic displacements and electric potential at point $\boldsymbol{x}(x_1,x_2,x_3)$ can be expressed as

$$\begin{cases} u_1=\sum_{i=1}^{3}A_i\lambda_i^u\dfrac{x_1-\xi_1}{R_iR_i^*} \\ u_2=\sum_{i=1}^{3}A_i\lambda_i^u\dfrac{x_2-\xi_2}{R_iR_i^*} \\ u_3=\sum_{i=1}^{3}A_i\lambda_i^w\dfrac{1}{R_i} \\ \phi=\sum_{i=1}^{3}A_i\lambda_i^\phi\dfrac{1}{R_i} \end{cases} \quad (28)$$

where

$$\begin{cases} R_i = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+z_i^2} \\ R_i^* = R_i + z_i \qquad\qquad i=0,1,2,3 \\ z_i = \nu_i(x_3-\xi_3) \end{cases} \tag{29}$$

$$\begin{cases} \lambda_i^u = [(c_{13}+c_{44})e_{33}-c_{33}(e_{15}+e_{31})]\nu_i^3 \\ \quad +(c_{44}e_{31}-c_{13}e_{15})\nu_i \\ \lambda_i^w = -c_{44}e_{33}\nu_i^4-[e_{31}(c_{13}+c_{44})-e_{33}c_{11} \\ \quad +e_{15}c_{13}]\nu_i^2-c_{11}e_{15} \\ \lambda_i^\phi = c_{33}c_{44}\nu_i^4+[c_{13}(c_{13}+2c_{44})-c_{11}c_{33}]\nu_i^2+c_{44}c_{11} \end{cases} \tag{30}$$

and A_i is determined by following equations

$$\begin{cases} \sum_{i=1}^{3} A_i\lambda_i^u = 0 \\ \sum_{i=1}^{3} A_i \dfrac{n_i^a}{\nu_i^2-1} = 0 \\ \sum_{i=1}^{3} A_i \dfrac{n_i^e}{\nu_i^2-1} = \dfrac{1}{2\pi} \end{cases} \tag{31}$$

here

$$\begin{cases} n_i^a = 2[\lambda_i^u(c_{13}+c_{44}\nu_i^2)+\nu_i\lambda_i^w(c_{44}-c_{33}) \\ \quad +\nu_i\lambda_i^\phi(e_{15}-e_{33})] \\ n_i^e = 2[-\lambda_i^u(e_{31}+e_{15}\nu_i^2)+\nu_i\lambda_i^w(e_{33}-e_{15}) \\ \quad +\nu_i\lambda_i^\phi(a_{11}-a_{33})] \end{cases} \tag{32}$$

4.1.2 Point force in x_3-direction For a unit point force in x_1-direction at point $\boldsymbol{\xi}(\xi_1,\xi_2,\xi_3)$, the elastic displacements and electric potential at point $\boldsymbol{x}(x_1,x_2,x_3)$ is expressed as

$$\begin{cases} u_1 = \sum_{i=1}^{3} B_i\lambda_i^u \dfrac{x_1-\xi_1}{R_iR_i^*} \\ u_2 = \sum_{i=1}^{3} B_i\lambda_i^u \dfrac{x_2-\xi_2}{R_iR_i^*} \\ u_3 = \sum_{i=1}^{3} B_i\lambda_i^w \dfrac{1}{R_i} \\ \phi = \sum_{i=1}^{3} B_i\lambda_i^\phi \dfrac{1}{R_i} \end{cases} \tag{33}$$

where B_i satisfies following equations

$$\begin{cases} \sum_{i=1}^{3} B_i\lambda_i^u = 0 \\ \sum_{i=1}^{3} B_i \dfrac{n_i^a}{\nu_i^2-1} = \dfrac{1}{2\pi} \\ \sum_{i=1}^{3} B_i \dfrac{n_i^e}{\nu_i^2-1} = 0 \end{cases} \tag{34}$$

4.1.3 Point force in x_1-direction For a unit point force in x_1-direction at point $\boldsymbol{\xi}(\xi_1,\xi_2,\xi_3)$, the elastic displacements and electric potential at point $\boldsymbol{x}(x_1,x_2,x_3)$ is expressed as

$$\begin{cases} u_1 = D_0\left[\dfrac{1}{R_0}-\dfrac{(x_2-\xi_2)^2}{R_0R_0^{*2}}\right]-\sum_{i=1}^{3} D_i\lambda_i^u\left[\dfrac{1}{R_i^*}-\dfrac{(x_1-\xi_1)^2}{R_iR_i^{*2}}\right] \\ u_2 = (x_1-\xi_1)(x_2-\xi_2)\left(D_0\dfrac{1}{R_0R_0^{*2}}+\sum_{i=1}^{3} D_i\lambda_i^u\dfrac{1}{R_iR_i^{*2}}\right) \\ u_3 = \sum_{i=1}^{3} D_i\lambda_i^w \dfrac{(x_1-\xi_1)}{R_iR_i^*} \\ \phi = \sum_{i=1}^{3} D_i\lambda_i^\phi \dfrac{(x_1-\xi_1)}{R_iR_i^*} \end{cases} \tag{35}$$

where D_i satisfies

$$\begin{cases} D_0\nu_0+\sum_{i=1}^{3} D_i\nu_i\lambda_i^u = 0 \\ \sum_{i=1}^{3} D_i\lambda_i^w = 0 \\ \sum_{i=1}^{3} D_i\lambda_i^\phi = 0 \\ D_0\nu_0c_{44}+\sum_{i=1}^{3} D_i\dfrac{n_i^t}{\nu_i^2-1} = \dfrac{1}{2\pi} \end{cases} \tag{36}$$

here

$$n_i^t = \nu_i\lambda_i^u(c_{44}-c_{11})+\lambda_i^w(c_{44}+c_{13}\nu_i^2)+\lambda_i^\phi(e_{15}+e_{31}\nu_i^2) \tag{37}$$

4.2 Hypersingular integral equations

For the transversely isotropic piezoelectric solid embedded a flat crack in the symmetric plane, the hypersingular integral Eq. (19) can be reduced to,

$$\oint_{S^+} \frac{1}{r^3}[c_{44}^2D_0\nu_0^2(2\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})+k_{11}(\delta_{\alpha\beta}-3r_{,\alpha}r_{,\beta})]\tilde{u}_\beta(Q)ds(Q) = -p_\alpha(P) \quad \alpha,\beta=1,2; \quad P\in S^+ \tag{38}$$

$$\oint_{S^+} \frac{k_{33}}{r^3}\tilde{u}_3(Q)ds(Q)+\oint_{S^+} \frac{k_{34}}{r^3}\tilde{\phi}(Q)ds(Q) = -p_3(P) \quad P\in S^+ \tag{39}$$

$$\oint_{S^+} \frac{k_{43}}{r^3}\tilde{u}_3(Q)ds(Q)+\oint_{S^+} \frac{k_{44}}{r^3}\tilde{\phi}(Q)ds(Q) = -q_0(P) \quad P\in S^+ \tag{40}$$

where $p_i(P)$ and $q_0(P)$ represent the mechanical and electrical loads on the crack surface due to internal or external loads, and they can obtained from the solution for the loads of the uncracked solid, and k_{IJ} is determined as

$$\begin{cases} k_{11}=\sum_{i=1}^{3}[c_{44}(A_i+D_i\nu_i)+e_{15}B_i] \\ \quad\times[c_{44}(\nu_i\lambda_i^u+\lambda_i^w)+e_{15}\lambda_i^\phi] \\ k_{33}=\sum_{i=1}^{3}(c_{33}A_i\nu_i+e_{33}B_i\nu_i-c_{13}D_i) \\ \quad\times(-c_{13}\lambda_i^u+c_{33}\nu_i\lambda_i^w+e_{33}\nu_i\lambda_i^\phi) \\ k_{34}=\sum_{i=1}^{3}(e_{33}A_i\nu_i-a_{33}B_i\nu_i-e_{31}D_i) \\ \quad\times(-c_{13}\lambda_i^u+c_{33}\nu_i\lambda_i^w+e_{33}\nu_i\lambda_i^\phi) \\ k_{43}=\sum_{i=1}^{3}(c_{33}A_i\nu_i+e_{33}B_i\nu_i-c_{13}D_i) \\ \quad\times(-e_{31}\lambda_i^u+e_{33}\nu_i\lambda_i^w-a_{33}\nu_i\lambda_i^\phi) \\ k_{44}=\sum_{i=1}^{3}(e_{33}A_i\nu_i-a_{33}B_i\nu_i-e_{31}D_i) \\ \quad\times(-e_{31}\lambda_i^u+e_{33}\nu_i\lambda_i^w-a_{33}\nu_i\lambda_i^\phi) \end{cases} \tag{41}$$

4.3 Singularity and singular stress and electric displacement field near the crack front

In order to investigate the singularity of the crack front, consider a local coordinate system defined as $x_1\ x_2\ x_3$. The x_1-axis is the tangent line of the crack front at point Q_0, x_2-axis is the internal normal line in the rack plane, and x_3 is the normal of the crack. Then the displacement and electric potential discontinuities of the crack surface near a crack front point Q_0 can be assumed as

$$\tilde{u}_k(Q)=g_k(Q_0)\xi_2^{\lambda_k}\quad \tilde{\phi}(Q)=\Phi(Q_0)\xi_2^{\lambda_4}\quad 0<\mathrm{Re}(\lambda_k)<1 \tag{42}$$

where $g_k(Q_0)$ and $\Phi(Q_0)$ are non-zero constants related to point Q_0, λ_K is the singular index at the crack front. Consider a small semi-circle domain S_ε on the crack surface including point Q_0 using the main-part analytical method[(26),(27)], the following relations can be derived

$$\oint_{S_\varepsilon}\frac{\tilde{u}_1}{r^3}d\xi_1 d\xi_2\cong-2\pi\lambda_1 g_1(Q_0)x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{43}$$

$$\oint_{S_\varepsilon}\frac{(x_1-\xi_1)^2}{r^5}\tilde{u}_1 d\xi_1 d\xi_2\cong-\frac{2}{3}\pi\lambda_1 g_1(Q_0)x_2^{\lambda_1-1}\cot(\lambda_1\pi) \tag{44}$$

$$\oint_{S_\varepsilon}\frac{(x_2-\xi_2)^2}{r^5}\tilde{u}_2 d\xi_1 d\xi_2\cong-\frac{4}{3}\pi\lambda_2 g_2(Q_0)x_2^{\lambda_2-1}\cot(\lambda_2\pi) \tag{45}$$

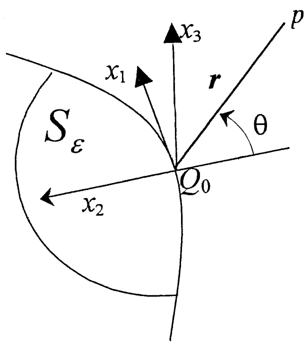

Fig. 1 A small semi-circle domain S_ε on the crack surface

$$\oint_{S_\varepsilon}\frac{(x_1-\xi_1)(x_2-\xi_2)}{r^5}\tilde{u}_1 d\xi_1 d\xi_2\cong 0 \tag{46}$$

$$\oint_{S_\varepsilon}\frac{\tilde{u}_3}{r^3}d\xi_1 d\xi_2\cong-2\pi\lambda_3 g_3(Q_0)x_2^{\lambda_3-1}\cot(\lambda_3\pi) \tag{47}$$

$$\oint_{S_\varepsilon}\frac{\tilde{\phi}}{r^3}d\xi_1 d\xi_2\cong-2\pi\lambda_4\Phi(Q_0)x_2^{\lambda_4-1}\cot(\lambda_4\pi) \tag{48}$$

Using above relations, from Eqs. (38)–(40), it can be shown that

$$\cot(\lambda_1\pi)=0\quad \cot(\lambda_2\pi)=0\quad \cot(\lambda_3\pi)=0\quad \cot(\lambda_4\pi)=0 \tag{49}$$

Then the singular indexes is obtained as

$$\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda=\frac{1}{2} \tag{50}$$

It is shown that the singularities near the crack front in a piezoelectric solid are the same as that in a general homogenous material. The mechanical stress intensity factors corresponding to the crack modes I, II and III as well as the “electric field intensity factor” K_{IV} are defined as

$$\begin{cases} K_I=\lim_{r\to0}\sigma_{33}(r,\theta)_{|\theta=0}\sqrt{2r}\quad K_{II}=\lim_{r\to0}\sigma_{32}(r,\theta)_{|\theta=0}\sqrt{2r} \\ K_{III}=\lim_{r\to0}\sigma_{31}(r,\theta)_{|\theta=0}\sqrt{2r}\quad K_{IV}=\lim_{r\to0}D_3(r,\theta)_{|\theta=0}\sqrt{2r} \end{cases} \tag{51}$$

where r is the distance from point p to the crack front point Q_0 as shown in Fig. 1. Considering relation Eq. (42), for a point p near the crack front, the following relations can be obtained by using the main-part analytical method.

$$\int_{S_\varepsilon}\frac{1}{R_0^3}\left(1-\frac{3(x_1-\xi_1)^2}{R_0^2}\right)\tilde{u}_1 d\xi_1 d\xi_2\cong 0 \tag{52}$$

$$\int_{S_\varepsilon}\frac{1}{R_0^3}\left[1-\frac{3(x_2-\xi_2)^2}{R_0^2}\right]\tilde{u}_1 d\xi_1 d\xi_2 \cong\frac{\pi g_1(Q_0)}{\sqrt{r}}\frac{1}{\sqrt{r_0}}\cos\frac{\theta_0}{2} \tag{53}$$

$$\int_{S_\varepsilon}\frac{1}{R_0^3}\left(1-\frac{3\nu_0^2x_3^2}{R_0^2}\right)\tilde{u}_1 d\xi_1 d\xi_2\cong\frac{\pi g_1(Q_0)}{\sqrt{r}}\frac{1}{\sqrt{r_0}}\cos\frac{\theta_0}{2} \tag{54}$$

$$\int_{S_\varepsilon}\frac{1}{R_j^3}\left[1-\frac{3(x_2-\xi_2)^2}{R_j^2}\right]\tilde{u}_3 d\xi_1 d\xi_2 \cong\frac{\pi g_3(Q_0)}{\sqrt{r}}\frac{1}{\sqrt{r_j}}\cos\frac{\theta_j}{2}\quad j=1,2,3 \tag{55}$$

$$\int_{S_\varepsilon}\frac{1}{R_j^3}\left(1-\frac{3\nu_j^2x_3^2}{R_j^2}\right)\tilde{u}_i d\xi_1 d\xi_2\cong\frac{\pi g_i(Q_0)}{\sqrt{r}}\frac{1}{\sqrt{r_j}}\cos\frac{\theta_j}{2} \tag{56}$$

$$\int_{S_\varepsilon}\frac{3\nu_j x_3(x_2-\xi_2)}{R_j^5}\tilde{u}_3 d\xi_1 d\xi_2\cong-\frac{\pi g_3(Q_0)}{\sqrt{r}}\frac{1}{\sqrt{r_j}}\sin\frac{\theta_j}{2} \tag{57}$$

where $r_0=\sqrt{\cos^2\theta+\nu_0^2\sin^2\theta}$, $\theta_0=\mathrm{tg}^{-1}(\nu_0\mathrm{tg}\theta)$, $r_je^{i\theta_j}=\cos\theta+i\nu_j\sin\theta$, and i is the imaginary unit $\sqrt{-1}$. Using relations Eqs. (52)–(57), the singular stresses and electric displacement around the crack front can be expressed as follows

$$\sigma_{13}=c_{44}^2 D_0 \nu_0^2 \frac{\pi g_1(Q_0)}{\sqrt{r}} \frac{1}{\sqrt{r_0}} \cos\frac{\theta_0}{2} \quad (58)$$

$$\sigma_{23}=-\frac{\pi g_2(Q_0)}{\sqrt{r}} \sum_{i=1}^{3} A_i^u \gamma_i^u \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} + \frac{\pi}{\sqrt{r}} \sum_{i=1}^{3} [g_3(Q_0)A_i^w + \Phi(Q_0)A_i^\phi]\gamma_i^u \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} \quad (59)$$

$$\sigma_{33}=-\frac{\pi g_2(Q_0)}{\sqrt{r}} \sum_{i=1}^{3} A_i^u \gamma_i^w \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} - \frac{\pi}{\sqrt{r}} \sum_{i=1}^{3} [g_3(Q_0)A_i^w + \Phi(Q_0)A_i^\phi]\gamma_i^w \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (60)$$

$$D_3=-\frac{\pi g_2(Q_0)}{\sqrt{r}} \sum_{i=1}^{3} A_i^u \gamma_i^\phi \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} - \frac{\pi}{\sqrt{r}} \sum_{i=1}^{3} [g_3(Q_0)A_i^w + \Phi(Q_0)A_i^\phi]\gamma_i^\phi \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (61)$$

here

$$\begin{cases} A_i^u = c_{44}(A_i + D_i \nu_i) + e_{15} B_i \\ A_i^w = c_{33} A_i \nu_i + e_{33} B_i \nu_i - c_{13} D_i \\ A_i^\phi = e_{33} A_i \nu_i - a_{33} B_i \nu_i - e_{31} D_i \end{cases} \quad (62)$$

$$\begin{cases} \gamma_i^u = c_{44}(\nu_i \lambda_i^u + \lambda_i^w) + e_{15} \lambda_i^\phi \\ \gamma_i^w = c_{13} \lambda_i^u - c_{33} \nu_i \lambda_i^w - e_{33} \nu_i \lambda_i^\phi \\ \gamma_i^\phi = e_{31} \lambda_i^u - e_{33} \nu_i \lambda_i^w + a_{33} \nu_i \lambda_i^\phi \end{cases} \quad (63)$$

Using relations Eq. (51), the above singular stresses and electric displacement can be rewritten as

$$\sigma_{13}=\frac{K_{III}}{\sqrt{2r}} \frac{1}{\sqrt{r_0}} \cos\frac{\theta_0}{2} \quad (64)$$

$$\sigma_{23}=\frac{K_I}{\sqrt{2r}} f_{21}(\theta) + \frac{K_{II}}{\sqrt{2r}} f_{22}(\theta) + \frac{K_{IV}}{\sqrt{2r}} f_{24}(\theta) \quad (65)$$

$$\sigma_{33}=\frac{K_I}{\sqrt{2r}} f_{31}(\theta) + \frac{K_{II}}{\sqrt{2r}} f_{32}(\theta) + \frac{K_{IV}}{\sqrt{2r}} f_{34}(\theta) \quad (66)$$

$$D_3=\frac{K_I}{\sqrt{2r}} f_{41}(\theta) + \frac{K_{II}}{\sqrt{2r}} f_{42}(\theta) + \frac{K_{IV}}{\sqrt{2r}} f_{44}(\theta) \quad (67)$$

where

$$f_{21}(\theta)=-\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{44}A_i^w - k_{43}A_i^\phi)\gamma_i^u \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} \quad (68)$$

$$f_{22}(\theta)=\frac{1}{k_{11}} \sum_{i=1}^{3} A_i^u \gamma_i^u \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (69)$$

$$f_{24}(\theta)=\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{33}A_i^w - k_{34}A_i^\phi)\gamma_i^u \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} \quad (70)$$

$$f_{31}(\theta)=\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{44}A_i^w - k_{43}A_i^\phi)\gamma_i^w \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (71)$$

$$f_{32}(\theta)=\frac{1}{k_{11}} \sum_{i=1}^{3} A_i^u \gamma \lambda_i^w \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} \quad (72)$$

$$f_{34}(\theta)=-\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{34}A_i^w - k_{33}A_i^\phi)\gamma_i^w \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (73)$$

$$f_{41}(\theta)=\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{44}A_i^w - k_{43}A_i^\phi)\gamma_i^\phi \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (74)$$

$$f_{42}(\theta)=\frac{1}{k_{11}} \sum_{i=1}^{3} A_i^u \gamma_i^\phi \frac{1}{\sqrt{r_i}} \sin\frac{\theta_i}{2} \quad (75)$$

$$f_{44}(\theta)=-\frac{1}{(k_{33}k_{44}-k_{34}k_{43})} \sum_{i=1}^{3} (k_{34}A_i^w - k_{33}A_i^\phi)\gamma_i^\phi \frac{1}{\sqrt{r_i}} \cos\frac{\theta_i}{2} \quad (76)$$

Other singular stresses and electric displacements near point Q_0 can also be obtained by use of above method. It is shown that, for the transversely isotropic piezoelectric materials, the crack mode I is not independent. That is, the mechanical stress intensity factor K_I is coupled with the "electric field intensity factor" K_{IV}.

5. Conclusion

A set of hypersingular integral equations coupled with general boundary integral equations for an impermeable crack in a three-dimensional piezoelectric solid subjected to mechanical and electrical loads is derived by a boundary element method. The unknowns are the discontinuities of the elastic displacements and electrical potential of the crack surface. The behaviors of the unknowns near the crack front are analyzed by the main-part analytical method of hypersingular singular integral equations, and the singular orders are given. It is shown that the singularities of the elastic stresses and electric displacement near the crack front in a piezoelectric solid are similar as that in a general homogenous material. Moreover, the singular stresses and electrical displacements near the crack front in a transversely isotropic piezoelectric solid can be obtained by main-part analytical method of two-dimensional hypersingular integral equations.

For an infinite transversely isotropic piezoelectric solid, it is shown that the crack mode II is coupled with mode III, and the crack mode I is also not independent. That is, the mechanical stress intensity factor $K_{\rm I}$ is coupled with the "electric field intensity factor" $K_{\rm IV}$.

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